CORRECTION TO "$H\mathbb{Z}$-ALGEBRA SPECTRA ARE DIFFERENTIAL GRADED ALGEBRAS"

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Abstract. This correction article is actually unnecessary. The proof of Theorem 1.2, concerning commutative $H\mathbb{Q}$-algebra spectra and commutative differential graded algebras, in the author's paper [*American Journal of Mathematics* **129** (2007) 351-379 (arxiv:math/0209215v4)] is correct as originally stated. Neil Strickland carefully proved that D is symmetric monoidal [St1]; so Proposition 4.7 and hence also Theorem 1.2 hold as stated. Strickland's proof will appear in joint work with Stefan Schwede [ScSt]; see related work in [arXiv:0810.1747] [St2]. Note here D is defined as a colimit of chain complexes; in contrast, non-symmetric monoidal functors analogous to D are defined as homotopy colimits of spaces in previous work of the author [S4].

We leave the old alternate approach to Theorem 1.2 below, with expository changes in the introduction, since it does provide another slightly weaker, non-natural statement.

In the author's paper [S1], the proof of Theorem 1.2 is correct as stated; the functor D is symmetric monoidal. The author's confusion about this fact came from the comparison of this functor D, which is defined as a colimit of chain complexes, with the functor D in [S4] which is defined as a homotopy colimit of spaces. See also the discussion of commutative I-monoids in section 2.2 of [Sc]. In the topological case D is not symmetric monoidal; in the algebraic case though the functor D is symmetric.

Since the paper [S1] is mainly concerned with associative algebras, the only place this issue arises is in the proof of Theorem 1.2. As stated in Remark 2.11 in [S1], the main theorems (Theorem 1.1, Corollary 2.15 and Corollary 2.16 in [S1]) would also hold with the "three step" functors H and Θ replaced by the "four step" functors $\overline{\mathbb{H}} = ULc\,\mathrm{C}_0\, fF_0c$ and $\overline{\Theta} = \mathrm{Ev}_0\, fi\phi^* NZc$ where c and f are the appropriate cofibrant and fibrant replacement functors. Since here the functors Ev_0, i, ϕ^*N and Z are symmetric monoidal, we have the following non-natural version of Theorem 1.2 from [S1] with Θ replaced by $\overline{\Theta}$. This statement first appeared as Theorem 1.3 in [S3].

Theorem 1. *For C any commutative $H\mathbb{Q}$-algebra, $\overline{\Theta}C$ is weakly equivalent to a commutative differential graded $\mathbb{Q}$-algebra.*

Proof. As noted in the proof of Theorem 1.2 from [S1], the reason $\overline{\Theta}$ is not symmetric monoidal is because the cofibrant and fibrant replacement functors involved in $\overline{\Theta}$ are not symmetric monoidal. This is why $\overline{\Theta}C$ is only weakly equivalent and not isomorphic to a commutative dg $\mathbb{Q}$ algebra.

The method for dealing with the cofibrant replacement functor in $\overline{\Theta}$ proceeds as in [S1]. As proved there, a natural zig-zag of weak equivalences exists between Zc and the symmetric monoidal functor $\alpha^*\widetilde{\mathbb{Q}}$. Let $\overline{\Theta}' = \mathrm{Ev}_0\, fi\phi^* N\alpha^*\widetilde{\mathbb{Q}}$. Then $\overline{\Theta}C$ is naturally weakly equivalent to $\overline{\Theta}'C$.

Next we need to consider the fibrant replacement functor f which appears in $\overline{\Theta}'$ (and $\overline{\Theta}$). This is the fibrant replacement functor in the model category of monoids in $Sp^{\Sigma}(\mathcal{C}h_{\mathbb{Q}})$. As in [S3], we exchange f for the fibrant replacement functor f' in the model category of commutative

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monoids in $Sp^\Sigma(\mathcal{C}h_\mathbb{Q})$ as established below in Proposition 3. For any commutative monoid A in $Sp^\Sigma(\mathcal{C}h_\mathbb{Q})$, we thus have two weak equivalences $A \longrightarrow fA$ and $A \longrightarrow f'A$. Since $f'A$ is also fibrant as a monoid and $A \longrightarrow fA$ is a trivial cofibration of monoids, lifting provides a weak equivalence $fA \longrightarrow f'A$. If we let $\overline{\Theta}''C = \mathrm{Ev}_0\, f' i\phi^* N\alpha^*\widetilde{\mathbb{Q}}$, we then have a (non-natural) weak equivalence $\overline{\Theta}'C \longrightarrow \overline{\Theta}''C$. Since $\overline{\Theta}C$ is weakly equivalent to $\overline{\Theta}'C$ and $\overline{\Theta}''C$ is a commutative differential graded $\mathbb{Q}$-algebra, this completes the proof. □

Remark 2. Although Theorem 1 does not give a natural identification of $\overline{\Theta}C$ with a commutative DGA, for any small, fixed I-diagram $\mathcal{D}$ of commutative $H\mathbb{Q}$-algebra spectra there will be a map of I-diagrams from $\overline{\Theta}\mathcal{D}$ to an I-diagram of commutative DGAs which is given by a variant of $\overline{\Theta}''\mathcal{D}$ with f' replaced by the fibrant replacement functor in the model category of I-diagrams of commutative monoids in $Sp^\Sigma(\mathcal{C}h_\mathbb{Q})$ given by [Hi, 11.6.1].

Proposition 3. *There is a model category structure on the category of commutative monoids in $Sp^\Sigma(\mathcal{C}h_\mathbb{Q})$ in which a map is a weak equivalence or fibration if and only if the underlying map in $Sp^\Sigma(\mathcal{C}h_\mathbb{Q})$ is so.*

Let $S_\mathbb{Q}$ denote the unit and let $\otimes_S$ denote the monoidal product in $Sp^\Sigma(\mathcal{C}h_\mathbb{Q})$. To establish this model category we use the lifting property from [ScSh, 2.3(i)] applied to the *free commutative monoid* functor $\mathbb{P}$ which is left adjoint to the forgetful functor from commutative monoids in $Sp^\Sigma(\mathcal{C}h_\mathbb{Q})$ to the underlying object in $Sp^\Sigma(\mathcal{C}h_\mathbb{Q})$. Namely, $\mathbb{P}(M) = \bigvee_{n\geq 0} M^{(n)}/\Sigma_n$ where $M^{(n)} = M \otimes_S \cdots \otimes_S M$ is the nth tensor power of M over $S_\mathbb{Q}$.

Let I denote the generating cofibrations and J denote the generating trivial cofibrations in $Sp^\Sigma(\mathcal{C}h_\mathbb{Q})$; see [Ho, 7]. To establish the lifting criterion in [ScSh, 2.3], we first show that applying $\mathbb{P}$ to any map in J produces a stable equivalence. We do this by showing that in the source and target the orbit constructions can be replaced by homotopy orbits without changing the homotopy type.

Lemma 4. *Let X, Y be in $Sp^\Sigma(\mathcal{C}h_\mathbb{Q})$ and $n \geq 1$.*

(1) *The map*
$$E\Sigma_n \otimes_{\Sigma_n} X^{(n)} \longrightarrow X^{(n)}/\Sigma_n$$
is a level equivalence.

(2) *The map*
$$(E\Sigma_n \otimes_{\Sigma_n} X^{(n)}) \otimes_S Y \longrightarrow (X^{(n)}/\Sigma_n) \otimes_S Y$$
is also a level equivalence.

Proof. The first statement follows directly from the fact that given any Σ_n-equivariant complex A in $\mathcal{C}h_\mathbb{Q}$, then
$$E\Sigma_n \otimes_{\Sigma_n} A \longrightarrow A/\Sigma_n$$
is a quasi-isomorphism. The second statement follows as well by extending the Σ_n-action trivially to Y and shifting the parentheses. □

Next we show that pushouts of maps in $\mathbb{P}(J)$ are stable equivalences and level cofibrations. Since directed colimits of such maps are again stable equivalences, Proposition 3 then follows from Lemmas 4 and 5 by [ScSh, 2.3].

Lemma 5. *Let $f : T \longrightarrow U$ be a cofibration in $Sp^\Sigma(\mathcal{C}h_\mathbb{Q})$ and V be a $\mathbb{P}T$-module. Then the map $q : V \longrightarrow V \otimes_{\mathbb{P}T} \mathbb{P}U$ is a level cofibration. If f is a trivial cofibration, then q is a stable equivalence.*

Proof. This follows from the fitration arguments of [Ma, 7.5, 8.6] using Lemma 4 instead of [Ma, 8.2, 8.10]. Note, here one does not need to restrict to the positive cofibrant objects since no such restriction is needed in Lemma 4. □

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